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# Localized fermions on quantized vortices in superfluid $^3$He-B

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**Abstract.** The quantized vortex in a superfluid or superconducting Fermi system my contain gapless fermionic excitations localized in the vortex core. The number of branches of the gapless excitations—fermionic zero modes—depends on the core structure and may change from 0 to $\varepsilon_F/\Delta \gg 1$. We investigated the possibility of the existence of the fermionic zero modes localized in the cores of axisymmetric v vortices in superfluid $^3$He-B. The zero modes exist if the core radius exceeds the coherence length.

## 1. Introduction

In Fermi superfluids and superconductors there are low-energy fermionic excitations localized in the cores of quantized vortices [1]. In conventional s wave superconductors the excitations localized near the vortex axis, where the phase of the order parameter has singularity and the modulus of the order parameter tends to zero, have a small energy gap of about $\Delta^2/\varepsilon_F \ll \Delta$ compared with the gap $\Delta$ of the delocalized excitation far from the vortex. As a result they play a decisive role in the kinetics and thermodynamics of the superconductor at low temperatures (see [2] for a review). Recently the density of the quasiparticle states in the vortex core has become an object of experimental investigations in scanning tunnelling microscopy experiments [3].

In modern relativistic field theories the vortices (strings) with the localized fermions are also discussed (see, e.g., [4, 5]). It is important that as distinct from the Abrikosov vortices in conventional superconductors, in the majority of the models for strings, one or more branches of fermions are the so-called fermionic zero modes, which means that they are gapless. As a rule the number of gapless fermions is related to the topological charge of the vortex (its winding number).

Since at a sufficiently low temperature $T \ll \Delta^2/\varepsilon_F$ the existence or absence of the gap in the localized fermion spectrum leads to essentially different behaviour, it is necessary to understand what are the conditions for the appearance of zero modes in the vortices in superconductors and superfluids.

As was shown in [6], the necessary condition for fermionic zero modes in condensed-matter vortices is the dissolving of the singularity on the vortex axis, i.e. the flaring out of the singularity from the vortex axis into extra dimensions, i.e. into momentum space. Such a phenomenon takes place for the quantized vortices in the superfluid $^3$He-B; the vortex with the singularity on the axis (o vortex) is unstable towards the formation of the v vortex with broken space parity which has no singularity on the vortex axis (see [7] for a review of quantized vortices in $^3$He). Therefore the order parameter is nowhere zero. This is not an exclusion; the doubly quantized Abrikosov vortex in a conventional superconductor also has no singularity on the axis [8] and one may expect instability of the singularity in singly quantized vortices in the superconductors with a complicated structure of the gap function in momentum space, such as heavy fermionic [9] and high-$T_c$ superconductors.

There are three levels of description of the quasiparticle spectrum in the quantized vortex. We are interested in the exact quantum mechanical spectrum $E_n(Q, k_z)$ which depends on two quantum numbers: the momentum $k_z$ along the vortex and the azimuthal quantum number $Q$. Zero modes are those branches of this spectrum which cross the zero energy level at some $k_z$. Sometimes the information on the existence of zero modes may be extracted from the behaviour of the classical energy spectrum $E(\boldsymbol{k}, \boldsymbol{r})$ which depends both on the momentum and on the position $\boldsymbol{r}$ inside the vortex core. In the simplest case this is the conventional equation $\sqrt{\varepsilon^2(\boldsymbol{k}) + |\Delta(\boldsymbol{k}, \boldsymbol{r})|^2}$, where the gap function $\Delta(\boldsymbol{k}, \boldsymbol{r})$ reflects the structure of the order parameter in the vortex core. Sometimes to investigate the zero modes it is necessary to know the behaviour of the spectrum of the intermediate level between classical and quantum levels; this is the semiclassical spectrum $E(Q, k_z, k_r, r)$ with quantized azimuthal motion but with the classical description of the radial motion.

The main difference between the singular and non-singular quantized vortices is in the behaviour of the classical spectrum of the fermionic excitations $E(\boldsymbol{k}, \boldsymbol{r})$. In the singular vortex the excitation energy $E(\boldsymbol{k}, \boldsymbol{r})$ drops to zero on the vortex axis ($r = 0$), where the gap function is zero, on the whole Fermi surface ($|\boldsymbol{k}| = k_F$). In the non-singular vortex with the gap function being non-zero on the vortex axis this two-dimensional surface of zeros is reoriented in the extended $(\boldsymbol{k}, \boldsymbol{r})$-space in the following manner: for each point $\boldsymbol{r}$ inside the definite radius $R$ (inner core radius) the energy drops to zero at several points $\boldsymbol{k} = \boldsymbol{k}^a(\boldsymbol{r})$ in momentum space. These are exclusive (so-called diabolical) points of the semiclassical spectrum, which are stable towards perturbation owing to conservation of their topological charge [10]. They are also known as 'boojums on the Fermi surface' (see [11]).

It appears that the number of the gapless fermionic modes $E_n(Q, k_z)$ on the vortices in the systems with Cooper pairing depends not on the winding number of the vortex but on the spatial distribution $\boldsymbol{k}^a(\boldsymbol{r})$, of zeros in the classical spectrum $E(\boldsymbol{k}, \boldsymbol{r})$, of the fermions in the vortex core. The number of fermionic zero modes may vary from zero to $\varepsilon_F/\Delta \gg 1$ for the same winding number of the vortex. For the so-called w vortex in superfluid $^3$He-A the existence of the fermionic zero modes was found [6] just from the distribution $\boldsymbol{k}^a(\boldsymbol{r})$ of classical zeros in spectrum. The number of zero modes proved to be of the order of $k_F R \gg 1$ where $R$ is the size of the non-singular core of this vortex. This is the result of the specific broken symmetry in the core of the w vortex which provides a non-zero spectral asymmetry index already in classical limit. This index gives rise to a large number of gapless branches of the quasiparticle spectrum $E_n(Q, k_z)$ and leads to spontaneous mass superflow along the vortex axis even at $T = 0$.

Here we consider the axisymmetric vortices in another superfluid phase of $^3$He. These vortices belong to the v type of symmetry which does not support the relevant topological index of spectral asymmetry on the classical level and therefore one needs to establish the existence of zero modes in this vortex in more detailed investigations on the intermediate semiclassical level ($E(Q, k_z, k_r, r)$). We found from the topological arguments that zero modes exist at least in the limit of the large core radius $R \gg \xi$. These arguments are illustrated on the simple model of the core structure of the axisymmetric v vortex with the A phase in the core. Our results suggest the existence of zero modes in real v vortices in $^3$He-B where $R = (5\text{–}10)\xi$.

## 2. Hamiltonian for fermions in the $^3$He-B vortex

The exact quantum mechanical spectrum of the fermions on the vortices is given by the eigenvalues of the Bogoliubov–Nambu Hamiltonian, describing the fermionic excitations:

$$\hat{H}(\boldsymbol{p}, \boldsymbol{r})\chi(\boldsymbol{r}) = E\chi(\boldsymbol{r})$$
$$\hat{H}(\boldsymbol{p}, \boldsymbol{r}) = \begin{pmatrix} \varepsilon(\boldsymbol{p}) & \Delta(\boldsymbol{p}) \\ \Delta^{\dagger}(\boldsymbol{p}) & -\varepsilon(\boldsymbol{p}) \end{pmatrix} \tag{2.1}$$

where $\boldsymbol{p} = (1/\mathrm{i})(\partial/\partial \boldsymbol{r})$ for the quantum problem, with $\varepsilon(\boldsymbol{p}) = (p^2 - k_F^2)/2m$ (where $k_F$ denotes the Fermi momentum).

The gap parameter $\boldsymbol{\Delta}(\boldsymbol{p}, \boldsymbol{r})$ is a symmetric $2 \times 2$ spin matrix for spin-triplet pairing and may be expressed in terms of the vector $\boldsymbol{d}$:

$$\Delta_{\alpha\beta}(\boldsymbol{p}, \boldsymbol{r}) = (g\boldsymbol{\sigma})_{\alpha\beta} \cdot \boldsymbol{d}(\boldsymbol{p}, \boldsymbol{r}) \tag{2.2}$$

where the $\boldsymbol{\sigma}$ abbreviate the Pauli spin matrices, and

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\mathrm{i}\boldsymbol{\sigma}_y$$

is the metric spinor.

For $L = 1$ pairing, $\boldsymbol{d}$ is linear in the momentum $\boldsymbol{p}$, and it may be expressed in terms of the order-parameter matrix $A_{\alpha i}$ as follows:

$$d_\alpha(\boldsymbol{p}, \boldsymbol{r}) = \tfrac{1}{2}[A_{\alpha i}(\boldsymbol{r})\, p_i/k_F + p_i/k_F]\, A_{\alpha i}(\boldsymbol{r}) \tag{2.3}$$

In the axisymmetric vortex with $m$ quanta of circulation the order parameter field is expressed in terms of nine radial functions $C_{\mu\nu}(r)$ (see review in [7]), which are amplitudes of the Cooper pairing with the spin projection $\mu = +1, 0, -1$ on the vortex axis and with the orbital momentum projection $\nu = +1, 0, -1$ on the same axis:

$$A_{\alpha i}^{\text{vortex}}(\boldsymbol{r}) = \Delta_B \sum_{\mu\nu} \lambda_\alpha^\mu \lambda_i^\nu C_{\mu\nu}(r) \exp[\mathrm{i}(m - \mu - \nu)\varphi]. \tag{2.4}$$

For the most symmetric o vortex, only five real $C_{\mu\nu}(r)$ are non-zero, with even $\mu + \nu$. In the v vortex which is stable at high pressures, all nine real components are non-zero while, for the w vortex, five components of the o vortex are real and the other four components are imaginary.

The fermionic excitations of the inhomogeneous vacuum of the vortex are described by the quantum numbers corresponding to the symmetry of the vacuum state. One of the quantum numbers is the projection $k_z$ of the quasiparticle momentum on the vortex axis. For the $m$-quantum axisymmetric vortex there is another continuous symmetry [7] which is the modification of conventional cylindrical symmetry described by the generator

$$\hat{Q} = \hat{L}_z + \hat{S}_z - m\hat{I}. \tag{2.5}$$

It differs from the generator of rotations about the axis of the cylindrical symmetry by the generator of the gauge transformation which transforms the order parameter in the following way:

$$\exp(i\hat{I}\Phi)\, A_{\alpha i} = \exp(i\Phi)\, A_{\alpha i}. \tag{2.6}$$

It is easy to check that for the vortex state in equation (2.4)

$$\hat{Q} A^{\text{vortex}}_{\alpha i} = 0.$$

Therefore the Fermi excitations as well as the bosonic collective modes of the oscillations of the order parameter in the vortex are described in addition to continuous quantum number $k_z$ by the quantum number $Q$. The generator $\hat{I}$ for the fermions is the Bogoliubov isospin $\frac{1}{2}\tau_3$ (the $\boldsymbol{\tau}$ are the Pauli matrices for the Bogoliubov isospin), i.e. $\frac{1}{2}$ for particle and $-\frac{1}{2}$ for the holes; therefore $Q$ is an integer for the vortices with odd quanta of circulation $m$ and is a half-integer for even $m$, just opposite to the case of the vortices in conventional s-wave superconductor with the spin-singlet pairing where $\hat{Q} = \hat{L}_z - m\hat{I}$.

The wavefunction of the Bogoliubov fermion with quantum numbers $k_z$ and $Q$ is as follows:

$$\chi(r) = \exp(ik_z z)\exp(iQ\varphi)\begin{bmatrix} u_1(r)\exp\{i[(m+1)/2]\varphi\} \\ u_2(r)\exp\{i[(m-1)/2]\varphi\} \\ u_3(r)\exp\{-i[(m+1)/2]\varphi\} \\ u_4(r)\exp\{-i[(m-1)/2]\varphi\} \end{bmatrix}. \tag{2.7}$$

The wavefunction $u(Q, k_z, r)$ satisfies the radial equation

$$\hat{H}(Q, k_z, r, -i\partial_r)u(r) = E(Q, k_z)u(r) \tag{2.8}$$

with the following matrix operator $\hat{H}$:

$$\hat{H}(Q, k_z, r, -i\partial_r) = \begin{bmatrix} a\left(\frac{m+1}{2}+Q\right) & 0 & b_-(Q) & -\frac{1}{\sqrt{2}}b_0(Q+\frac{1}{2}) \\ 0 & a\left(\frac{m-1}{2}+Q\right) & -\frac{1}{\sqrt{2}}b_0(Q-\frac{1}{2}) & -b_+(Q) \\ b_-(-Q) & -\frac{1}{\sqrt{2}}b_0(-Q+\frac{1}{2}) & -a\left(\frac{m+1}{2}-Q\right) & 0 \\ -\frac{1}{\sqrt{2}}b_0(-Q-\frac{1}{2}) & -b_+(-Q) & 0 & -a\left(\frac{m-1}{2}-Q\right) \end{bmatrix} \tag{2.9}$$

where

$$a(l) = \tfrac{1}{2}[-(\partial_r)^2 - (1/r)\partial_r + l^2/r^2 + k_z^2 - k_F^2] \tag{2.10a}$$

$$b_\mu(l, k_z)/\Delta_B = -(i/2)[(C_{\mu+} + C_{\mu-})\partial_r + \partial_r(C_{\mu+} + C_{\mu-})] + i\{[(l-\tfrac{1}{2})/r]C_{\mu+} - [(l+\tfrac{1}{2})/r]C_{\mu-}\} + \sqrt{2}C_{\mu 0}k_z. \tag{2.10b}$$

The Hamiltonian (2.9) for the excitations in the core of the v vortex has the following elements of symmetry:

$$\hat{H}(-Q, k_z) = -\tau_2\hat{H}(Q, k_z)\tau_2 \tag{2.11a}$$

$$\hat{H}(Q, -k_z) = \tau_3\hat{H}^*(Q, k_z)\tau_3 \tag{2.11b}$$

$$\hat{H}(-Q, -k_z) = -\tau_1 \hat{H}^*(Q, k_z)\tau_1. \tag{2.11c}$$

The last one corresponds to the general '*CPT* theorem' for the Bogoliubov excitations in the axisymmetric vortex. The second one results from the additional discrete symmetry of the vortex; here this is the so-called v symmetry of the v vortex [7], which is the combined symmetry $PTU_2$: the symmetry under simultaneous space inversion $P$, time inversion $T$ and rotation $U_2$ by $\pi$ about the axis which is perpendicular to the vortex axis ($PTU_2 k_z = -k_z$, $PTU_2 Q = Q$). Equation (2.11*a*) is the combination of others. Since $CPT\, E(Q, k_z) = -E(-Q, -k_z)$ and $PTU_2\, E(Q, k_z) = E(Q, -k_z)$, equations (2.11) lead to the following symmetry of the spectrum:

$$E(Q, k_z) = E(Q, -k_z) = -E(-Q, -k_z). \tag{2.12}$$

In the most symmetric o vortex the symmetries $P$ and $TU_2$ are conserved separately ($\hat{H}(Q, -k_z) = \sigma_3 \hat{H}(Q, k_z)\sigma_3$ and $\hat{H}(Q, k_z) = \sigma_3\tau_3\hat{H}^*(Q, k_z)\tau_3\sigma_3$ correspondingly); however, this does not produce an additional symmetry of the spectrum, since the $TU_2$ symmetry operation does not transform $k_z$ and $Q$: $TU_2\, E(Q, k_z) = E(Q, k_z)$. In the w vortex the $TU_2$ symmetry exists while the $P$ and $PTU_2$ symmetries are broken. This difference in symmetry results in a different behaviour of the quasiparticle spectrum in v and w vortices; zero modes always exist in the w vortex but there should be a special condition for the existence of zero modes in the v vortex.

Let us make the following substitution for the wavefunction $u$ to make the matrix $\hat{H}$ Hermitian:

$$u(r) = (1/\sqrt{r})\tilde{u}(r) \tag{2.13}$$

and neglect all the terms of order $\Delta^2/\varepsilon_F$. Then the corresponding Hamiltonian acting on $\tilde{u}$ is

$$\hat{H}(Q, k_z, r, -\mathrm{i}\partial_r)$$
$$= \begin{bmatrix} a\left(\frac{m+1}{2}+Q\right) & 0 & b_-(Q) & -\frac{1}{\sqrt{2}}b_0(Q+\frac{1}{2}) \\ 0 & a\left(\frac{m-1}{2}+Q\right) & -\frac{1}{\sqrt{2}}b_0(Q-\frac{1}{2}) & -b_+(Q) \\ b_-(-Q) & -\frac{1}{\sqrt{2}}b_0(-Q+\frac{1}{2}) & -a\left(\frac{m+1}{2}-Q\right) & 0 \\ -\frac{1}{\sqrt{2}}b_0(-Q-\frac{1}{2}) & -b_+(-Q) & 0 & -a\left(\frac{m-1}{2}-Q\right) \end{bmatrix} \tag{2.14}$$

where

$$a(l) = \tfrac{1}{2}[-(\partial_r)^2 + (l^2 - \tfrac{1}{4})/r^2 + k_z^2 - k_F^2] \tag{2.15a}$$

$$b_\mu(l, k_z)/\Delta_B = -(\mathrm{i}/2)[(C_{\mu+} + C_{\mu-})\partial_r + \partial_r(C_{\mu+} + C_{\mu-})]$$
$$+ \mathrm{i}(l/r)(C_{\mu+} - C_{\mu-}) + \sqrt{2}C_{\mu 0}k_z. \tag{2.15b}$$

## 3. Semiclassical approach to zero-mode problem

It is impossible to find analytically the exact quantum mechanical spectrum $E_n(Q, k_z)$ of the Hamiltonian (2.14). However, to investigate the possibility of the existence of fermionic zero modes, i.e. branches of the spectrum $E_n(Q, k_z)$ which intersect the zero energy at some $k_z = k_z^0$, it is not necessary to solve the eigenvalue problem. This can be done using the semiclassical approximation which is good since the wavelength $k_F^{-1}$ of excitations is much less than the characteristic scale of the potential created by the vortex which is of order of coherence length $\xi = v_F/\Delta_B$. In this limit the radial momentum operator may be substituted by its classical value

$$-\mathrm{i}\partial_r \to k_r. \tag{3.1}$$

In this case the Hamiltonian $\hat{H}(Q, k_z, r, -\mathrm{i}\partial_r)$ transforms to the matrix $H(Q, k_z, r, k_r)$.

As will be seen further, the zero modes take place with large $Q \gg 1$. In this limit the semiclassical matrix operator $H$ has the following form:

$$H(Q, k_z, r, k_r) = \begin{bmatrix} \varepsilon & 0 & b_- & -(1/\sqrt{2})b_0 \\ 0 & \varepsilon & -(1/\sqrt{2})b_0 & -b_+ \\ b_-^* & -(1/\sqrt{2})b_0^* & -\varepsilon & 0 \\ -(1/\sqrt{2})b_0^* & -b_+^* & 0 & -\varepsilon \end{bmatrix} \tag{3.2}$$

where

$$\varepsilon = \tfrac{1}{2}(k_r^2 + Q^2/r^2 + k_z^2 - k_F^2) \tag{3.3}$$

$$b_\mu/\Delta_B = C_{\mu+}(r)k_+ + C_{\mu-}(r)k_- + \sqrt{2}C_{\mu 0}(r)k_z \tag{3.4}$$

and $k_\pm = k_r \pm \mathrm{i}Q/r$.

This matrix may be obtained from the pure classical Bogoliubov matrix for the excitations in the non-uniform order parameter field of the vortex, i.e.

$$H(\boldsymbol{k}, \boldsymbol{r}) = \begin{pmatrix} \varepsilon(\boldsymbol{k}) & \Delta(\boldsymbol{r}, \boldsymbol{k}) \\ \Delta^\dagger(\boldsymbol{r}, \boldsymbol{k}) & -\varepsilon(\boldsymbol{k}) \end{pmatrix} \tag{3.5}$$

if one introduces two constraints on the five variables $\boldsymbol{k}, \boldsymbol{r} = (x, y)$ (or $(r, \varphi)$ in cylindrical coordinate system) which correspond to the quantization of the azimuthal motion: the $\varphi$ component of the momentum $\boldsymbol{k} = (k_z, k_r, k_\varphi)$ is $k_\varphi = Q/r$ and the $\varphi$ component of the coordinate is $\varphi = 0$. We begin first with the properties of the classical spectrum $E(\boldsymbol{k}, \boldsymbol{r})$ of the matrix $H(\boldsymbol{k}, \boldsymbol{r})$.

## 4. Diabolical points in classical and semiclassical spectrum

The important property of the eigenvalues $E(\boldsymbol{k}, \boldsymbol{r})$ of the classical matrix in equation (3.5), i.e. of the classical spectrum of the quasiparticles in the vortex core, is the existence of zeros in the spectrum [7]; for each point $\boldsymbol{r} = (x, y)$ inside some radius $r_{\text{core}}$ near the vortex axis there exist several points in momentum space $\boldsymbol{k} = \boldsymbol{k}^a(\boldsymbol{r})$ at which the quasiparticle has zero energy, i.e. $E(\boldsymbol{k}^a(\boldsymbol{r}), \boldsymbol{r}) = 0$. These points in momentum space (so-called 'boojums on the Fermi surface') are diabolical points, i.e. exclusive points of intersection of different branches of spectrum. Here the quasiparticle branch $E(\boldsymbol{k}, \boldsymbol{r})$ of the quasiclassical spectrum touches the quasihole branch $-E(\boldsymbol{k}, \boldsymbol{r})$ of spectrum and therefore the classical spectrum $E(\boldsymbol{k}, \boldsymbol{r})$ touches the zero energy level. One may expect that the quantum numbers $Q$ and $k_z$ in exact quantum mechanical spectrum $E(Q, k_z)$

should be close to the corresponding classical values near diabolical point to obtain zero in exact spectrum $E(Q, k_z)$.

The diabolical point has topological characteristics due to which it is stable and does not disappear under perturbations. The topological invariant describing the point $\boldsymbol{k}^a(\boldsymbol{r})$ may be written in terms of the classical Green function [10]

$$G = (\mathrm{i}\omega - H)^{-1} \tag{4.1}$$

as the integral over three-dimensional surface $\sigma$ around point $\omega = 0$, $\boldsymbol{k} = \boldsymbol{k}^a(\boldsymbol{r})$ in four-dimensional $(\boldsymbol{k}, \omega)$-space:

$$N^a = \frac{1}{24\pi^2} \mathrm{e}^{ijkl} \operatorname{Tr} \int_\sigma \mathrm{d}S_i\, G\partial_j G^{-1} G\partial_k G^{-1} G\partial_l G^{-1}. \tag{4.2}$$

This is an integer-valued invariant which equals +1 or −1 for diabolical points inside the vortex core.

Let us consider the positions of the diabolical points on the simple example of the $^3$He-B v vortex which consists of the B phase outside the core and the A phase inside the core; all the amplitudes $C_{\mu\nu}$ are zero with the exception of the B-phase amplitudes and the A-phase amplitude concentrated in the core of large radius $R$:

$$C_{+-} = C_{-+} = C_{00} = B(r)/\Delta_\mathrm{B} \qquad C_{0+} = \sqrt{2}A(r)/\Delta_\mathrm{B} \tag{4.3}$$

with $B(r)/\Delta_\mathrm{B} \to r/R$ at $r \ll R$ and $B(r \gg R) = \Delta_\mathrm{B}$, $A(0) = \Delta_\mathrm{B}$ and $A(r \gg R) = 0$, and $R \gg \xi$.

The positions of zeros of the classical energy spectrum $E(\boldsymbol{k}, \boldsymbol{r})$ for this simple vortex

$$\boldsymbol{k}^a\,(r, \varphi) = \hat{z}\cos\beta(r) + \sin\beta(r)\,[\hat{r}\cos\alpha(r) + \hat{\varphi}\sin\alpha(r)] \tag{4.4}$$

is given here by equations

$$\tan^2\beta(r) = \tan^2\alpha(r) = B^2(r)/[A^2(r) - B^2(r)]. \tag{4.5}$$

Now we can proceed to the semiclassical spectrum $E(Q, k_z, k_r, r)$ of the semiclassical matrix in equation (3.2). The Hamiltonian in equation (3.5) and that in equation (3.2) depend on a different number of variables: five variables $(\boldsymbol{k}, \boldsymbol{r})$ in equation (3.5) while the number of variables in equation (3.2) is reduced to $k_z$, $k_r$, $r$ owing to the quantization of the azimuthal motion in terms of $Q$, which eliminated the variables $\varphi$ and $k_\varphi$. Before the quantization had been done the manifold of zeros of the classical Hamiltonian (3.5) forms two-dimensional subspace in the five-dimensional $(\boldsymbol{k}, x, y)$-space, since for each $\boldsymbol{r}$ inside the core one has point zeros in momentum $\boldsymbol{k}$-space. Now, when the axial degree of freedom is quantized in terms of $Q$ eliminating the two variables $\varphi$ and $k_\varphi$, this manifold of zeros reduces to points $k_z^{(0)}, k_r^{(0)}, r^{(0)}$ in the three-dimensional $(k_z, k_r, r)$-space of the semiclassical spectrum $E(Q, k_z, k_r, r)$. These points are also the diabolical points, now in mixed momentum and coordinate space.

For given $Q$ the points $k_z^{(0)}$, $k_r^{(0)}$, $r^{(0)}$ may be found from equations (4.4) and (4.5) if one takes into account that

$$\tan\alpha = k_\varphi/k_r = Q/rk_r \qquad \tan^2\beta = (k_r^2 + Q^2/r^2)/k_z^2. \tag{4.6}$$

In the range of quantum numbers $Q$ where

$$1 \ll Q \ll k_F R \tag{4.7}$$

equations (4.5) and (4.6) have two solutions; for positive and negative $k_z$, they are related by symmetry relations. For positive $k_z$, one has

$$\begin{aligned} &r^{(0)} = R(Q/k_F R)^{1/3} \ll R \qquad k_r^{(0)} = k_F(r^{(0)}/R) \ll k_F \\ &k_\varphi^{(0)} = k_F(r^{(0)}/R)^2 \ll k_r^{(0)} \ll |k_z^{(0)}| \simeq k_F \qquad k_z^{(0)} = k_F[1 - \tfrac{1}{2}(Q/k_F R)^{2/3}]. \end{aligned} \tag{4.8}$$

This point is described by the same topological invariant (4.2) but the integration takes place in the four-dimensional $(\omega, k_z, k_r, r)$-space.

## 5. Diabolical points and spectral asymmetry index

It is important that the diabolical points of both Hamiltonians in equations (3.2) and (3.5) take place exactly at zero energy level. Therefore the diabolical point of the quasiclassical spectrum is simultaneously the zero in the spectrum. This follows from the additional symmetry of the quasiclassical Hamiltonians (3.2) and (3.5):

$$H^* = -\tau_2 H \tau_2 \tag{5.1}$$

which implies that, if $E(\boldsymbol{k}, \boldsymbol{r})$ is the eigenvalue of this matrix, then $-E(\boldsymbol{k}, \boldsymbol{r})$ is also the eigenvalue. Therefore, if the spectrum $E(\boldsymbol{k}, \boldsymbol{r})$ of the Hamiltonian (3.5) or the spectrum $E(k_z, k_r, r)$ of the Hamiltonian (3.2) touches zero, it means that the positive and negative branches of spectrum touch each other and one has the diabolical point of the spectrum, an exclusive point of intersection of the branches of the quasiparticle spectrum.

This symmetry is, however, approximate for the semiclassical Hamiltonian and takes place only in the limit of large $Q$ where the semiclassical Hamiltonian transforms to the classical Hamiltonian. Let us now introduce terms of the relative order of $1/Q$ which were neglected in equation (3.2). As follows from the exact Hamiltonian (2.14) the main term of this type is

$$(1 + \sigma_3)Q/2r^2. \tag{5.2}$$

This term gives the non-zero trace of the Bogoliubov matrix, i.e.

$$\mathrm{Tr}\,\hat{H} = 2Q/r^2$$

and violates the symmetry in equation (5.1). As a result it moves the diabolical point of the intersection of the branches of the semiclassical spectrum from the zero energy level, i.e. the branches of the semiclassical quasiparticle spectrum $E(k_z, k_r, r)$ now cross each other at finite energy of the order of $Q/r_0^2$. This means that the lower branch (the branch below the diabolical point) may cross the zero level on the whole two-dimensional surface in the three-dimensional $(k_z, k_r, r)$-space.

Now we show that the existence of zero modes in the exact quantum mechanical spectrum $E(Q, k_z)$ depends on the topology and size of this two-dimensional surface. Let us suppose that the lower branch of the $E(k_z, k_r, r)$ spectrum below the diabolical point with positive $k_z$ intersects zero level and all the two-dimensional surface of zeros lies in the half-space $k_z > 0$ in the region $0 < k_{z1} < k_z < k_{z2}$. This surface is compact and is characterized by the same topological invariant as the diabolical point, where the integral now is around the surface $\sigma$ embracing the surface of zeros in four-dimensional $(\omega, k_z, k_r, r)$-space:

$$N = \frac{1}{24\pi^2} e^{ijkl} \operatorname{Tr} \left( \int_\sigma dS_i \, G\partial_j G^{-1} G\partial_k G^{-1} G\partial_l G^{-1} \right). \tag{5.3}$$

Here $G$ is the Green function of the semiclassical Hamiltonian:

$$G(Q, \omega, k_z, k_r, r) = [i\omega - H(Q, k_z, k_r, r)]^{-1}. \tag{5.4}$$

Now one may show that, in this case, at least one branch of the exact spectrum $E(Q, k_z)$ crosses zero at some $k_{z0}$ inside the segment $k_{z1} < k_{z0} < k_{z2}$. To prove this let us introduce the spectral asymmetry index $N(k_z)$ (see, e.g., [5]) which may be expressed in terms of the Green function of the exact Hamiltonian (2.14):

$$\hat{G} = (i\omega - \hat{H})^{-1} \tag{5.5}$$

$$N(Q, k_z) = \operatorname{Tr} \left( \int \frac{d\omega}{2\pi} \hat{G} \right) = -\operatorname{Tr} \int \frac{d\omega}{2\pi} \frac{\hat{H}}{\omega^2 + \hat{H}^2} \Big) = -\tfrac{1}{2} \sum_n \operatorname{sgn} E_n(Q, k_z). \tag{5.6}$$

Here Tr means the summation over all the states with given $Q$ and $k_z$. This integer-valued index shows the difference between the numbers of the positive and negative eigenvalues $E_n(Q, k_z)$ of $\hat{H}$ at a given $Q$ and momentum $k_z$, if the index $N(k_z)$ changes abruptly at some $k_z$, this means that at this $k_z$ one of the energy levels $E_n(Q, k_z)$ crosses zero energy.

Since the core size is of the order of the coherence length $\xi = v_F/\Delta$ and therefore is much larger than the wavelength $k_F^{-1}$ of the excitations, one may use the gradient expansion for the Green function $\hat{G}$ (see, e.g., [11]) and express $\hat{G}$ in terms of the semiclassical Green function in equation (5.4):

$$\begin{aligned} \hat{G}(\omega, Q, k_z, -i\partial_r, r) &= G(\omega, Q, k_z, k_r, r) \\ &+ (i/2) G(\partial_{k_r} G^{-1} G\partial_r G^{-1} G - \partial_r G^{-1} G\partial_{k_r} G^{-1} G) + \dots . \end{aligned} \tag{5.7}$$

Now substituting equation (5.7) into equation (5.6) and using the equations $\partial_\omega G^{-1} = i$ and

$$\operatorname{Tr} = \int \frac{dk_r \, dr}{2\pi} \operatorname{tr}$$

where tr means trace only for matrix indices, one has the following expression for spectral asymmetry index:

$$N(Q, k_z) = \frac{1}{24\pi^2} e^{jkl} \operatorname{tr} \left( \int d^3x \, G\partial_j G^{-1} G\partial_k G^{-1} G\partial_l G^{-1} \right) \tag{5.8}$$

where $(x_1, x_2, x_3) = (\omega, r, k_r)$. This equation is well defined only if the Green function has no singularities, i.e. if the region of integration does not contain the zeros in the semiclassical spectrum $E(Q, k_z, k_r, r)$.

Equation (5.8) allows us to relate the number of zero modes with the topological charge $N$ in equation (5.3). Let us consider the difference of asymmetry indices at different points $k_z$ on both sides of the region of zeros in the semiclassical spectrum where the index is well defined:

$$N(Q, k_z > k_{z2}) - N(Q, k_z < k_{z1}).$$

On the one hand the difference between the two integrals coincides with the integral $N$ in equation (5.4) around the manifold of zeros in the semiclassical spectrum; on the other hand this difference is just the number of branches of exact spectrum $E_n(Q, k_z)$ which cross zero energy level. Therefore, since $N = \pm 1$, then at least one zero mode exists in the spectrum.

The number of zero modes in principle may be larger since $N$ gives the algebraic sum of zero modes; it does not count the branches which twice intersect the zero level. One can also estimate the total number of the zero modes in the exact quasiparticle spectrum $E_n(Q, k_z)$ using the Bohr quantization rule. At given $k_z$ the manifold of zeros in the semiclassical spectrum is the closed line in the two-dimensional $(k_r, r)$ phase space at which the quasiclassical energy is zero. One must estimate the area

$$\oint k_r \, \mathrm{d}r$$

inside this curve and compare with the elementary quantum $2\pi\hbar$; their ratio gives the number of the positive energy levels of the lower branch and therefore shows the spectral asymmetry

$$N(Q, k_z) = \oint \frac{k_r \, \mathrm{d}r}{2\pi\hbar} \tag{5.9}$$

for arbitrary $k_z$ even inside the segment $k_{z1} < k_z < k_{z2}$. If the area in equation (5.9) is large, the $N(Q, k_z)$ may be calculated in the quasiclassical approximation, where the integer-valuedness of this index is restored owing to Bohr quantization rule. In this case since the spectral asymmetry is absent at $k_z > k_{z2}$ the $N(Q, k_z)$ shows how many branches $E_n(Q, k_z)$ of the exact quasiparticle spectrum crossed zero level inside the segment $(k_z, k_{z2})$.

## 6. Zero modes in the simple model of the vortex

Now let us apply this to the simple model (4.3) of the v vortex which qualitatively represents the real v vortex in the superfluid $^3$He-B. The Bogoliubov semiclassical matrix for the semiclassical spectrum $E(Q, k_z, r, k_r)$ of fermions in this vortex including the centrifugal term $(1 + \sigma_3)Q/2r^2$ is

$$H(Q, k_z, r, k_r) = \tilde{H}(Q, k_z, r, k_r) + (1 + \sigma_3)Q/2r^2$$

$$\tilde{H}(Q, k_z, r, k_r) = \begin{bmatrix} \varepsilon & 0 & B(r)k_+ & -A(r)k_+ - B(r)k_z \\ 0 & \varepsilon & -A(r)k_+ - B(r)k_z & -B(r)k_- \\ B(r)k_- & -A(r)k_- - B(r)k_z & -\varepsilon & 0 \\ -A(r)k_- - B(r)k_z & -B(r)k_+ & 0 & -\varepsilon \end{bmatrix} \tag{6.1}$$

where $k_\pm = k_r \pm iQ/r$. The manifold of zeros of the eigenvalues $E(Q, k_z, r, k_r)$ of this matrix is obtained as zeros of the determinant of the matrix

$$\det H = [\varepsilon^2 + (B\sqrt{k_r^2 + Q^2/r^2} - AQ/r)^2 + (Ak_r + Bk_z)^2] \times [\varepsilon^2 + (B\sqrt{k_r^2 + Q^2/r^2} - AQ/r)^2 + (Ak_r + Bk_z)^2] - (\varepsilon^2 + B^2k_\perp^2)Q^2/r^4. \tag{6.2}$$

This determinant is positive at $k_z > k_F$ for all $k_r$ and $r$ and therefore has no zeros at large $k_z$. Zeros appear at lower $k_z$. According to equations (4.6) and (4.7) for $Q \ll k_F R$ there is a point $k_z^{(0)}$, $k_r^{(0)}$, $r^{(0)}$ at which

$$\varepsilon = Ak_r + Bk_z = B\sqrt{k_r^2 + Q^2/r^2} - AQ/r = 0$$

which means that the determinant is negative at this point and therefore there are zeros in the semiclassical quasiparticle spectrum $E(Q, k_z, r, k_r)$ at $k_z = k_z^{(0)}$. Then it may be shown that at $k_z = 0$ the determinant is again always positive if $Q \gg k_F\xi$. The same situation takes place for the negative $k_z$. Therefore in the range of $Q$ where

$$k_F\xi \ll Q \ll k_F R \tag{6.3}$$

the manifold of zeros in the semiclassical spectrum $E(Q, k_z, r, k_r)$ consists of two isolated compact manifolds: one is concentrated in the positive half-space $k_z > 0$ and the other in the negative half-space. Both manifolds arise from the diabolical points and therefore have non-zero topological invariants $N = \pm 1$.

According to section 5 this means that there exist at least two branches $E_n(Q, k_z)$ for each azimuthal quantum number $Q$ in the range (6.3), which intersect zero energy level; one branch intersects zero at positive $k_z$ while the other intersects zero at negative $k_z$. To have the corresponding $Q$-values the core radius $R$ should essentially exceed the coherence length $\xi$. This is believed to be the case for the $^3$He-B vortices (see [7]). The total number of zero modes is of order $k_F R \sim \varepsilon_F/\Delta_B$.

## 7. Discussion

As distinct from the traditional singular vortices the vortex with a dissolved singularity may have the gapless fermionic excitations. As distinct from the fermions on strings in elementary-particle theories, the number of the fermionic zero modes (number of 'families' of massless chiral fermions) on the vortices in the pair-correlated systems is not defined by the winding number of the vortex but is of order of $k_F R$, where $R$ is the width of the dissolved singularity on the axis, i.e. the size of the region inside the vortex core where the diabolical points of the fermionic quasiclassical spectrum $E(\boldsymbol{k}, \boldsymbol{r})$ are concentrated.

The fermionic excitations on the vortices are characterized by the integer azimuthal quantum number $Q$ resulting from the continuous $Q$-symmetry of the axisymmetric vortex. While for the vortices with w symmetry the fermionic zero modes always exist, we found here that for the v vortices the zero modes exist only for $k_F\xi < Q < k_F R$ which means that the core radius should exceed the coherence length for existence of zero modes.

The number of 'families' of gapless fermions is an additional characteristic of the vortices of given winding number and of given symmetry class. The transition between vortices of the same topology and symmetry class but with different numbers of fermionic zero modes is the analogue of the Lifshitz [12] transition of the $2\frac{1}{2}$ order at $T = 0$ at which the topology of the Fermi surface changes; in the system of the one-dimensional fermions localized on the vortices, the zero-dimensional Fermi 'surface' appears or disappears. Thus we have the following hierarchy of characteristics of the vortices or other topological objects: topological charges of the vortex → symmetry of the vortex → topology of the Fermi surface of the gapless excitations inside the vortex core.

At temperatures well below $\Delta^2/\varepsilon_F$, instability of the one-dimensional Fermi system of localized Bogoliubov excitations towards either Cooper pairing or formation of the spin-density wave should develop. The type of instability depends on the sign of the coupling constant between the one-dimensional fermions and may be different for different families of fermions. As a result the additional breaking of symmetry in the vortex core is expected at low temperatures giving rise to such phenomena as periodic modulation of the order parameter along the vortex axis, spiral texture, s-wave superfluidity or other superfluidity classes.

Note that the flaring out of the singularities from the vortex axis into higher dimensions, which gives rise to the fermionic zero modes on vortices, occurs not only in systems with a multicomponent order parameter such as superfluid $^3$He and possibly heavy-fermion superconductors. Singularities are also dissolved inside the vortices in conventional superconductors: in a doubly quantized Abrikosov vortex [8] and even in singly quantized vortices where the non-singular four-particle-correlated states appear on the vortex axis [13]. Therefore the fermionic zero modes may exist even in conventional superconductors, especially if the vortex core structure becomes complicated owing to crystal-field anisotropy. The number of branches of gapless fermions in this case should vary from 0 to $\varepsilon_F/\Delta \gg 1$. It is possible that new experiments with a scanning tunnelling microscope [3] could reveal this fine structure of the Abrikosov vortices in superconductors.

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**References**

[1] Caroli C, De Gennes P G and Matricon J 1964 *Phys. Lett.* **9** 307
[2] Gor'kov L P and Kopnin N B 1975 *Usp. Fiz. Nauk* **116** 413
[3] Hess H F, Robinson R B, Dynes R C, Valles J M Jr and Waszczak J V 1989 *Phys. Rev. Lett.* **62** 214
[4] Witten E 1985 *Nucl. Phys.* B **249** 557
[5] Widrow L M 1988 *Phys. Rev.* D **38** 1684
[6] Volovik G E 1989 *Zh. Eksp. Teor. Fiz. Pis. Red.* **49** 343
[7] Salomaa M M and Volovik G E 1987 *Rev. Mod. Phys.* **59** 533
[8] Volovik G E 1988 *J. Phys. C: Solid State Phys.* **21** L221
[9] Schenstrom A *et al* 1989 *Phys. Rev. Lett.* **62** 332
[10] Grinevich P G and Volovik G E 1988 *J. Low Temp. Phys.* **72** 371
[11] Volovik G E and Mineev V P 1982 *Pis. Zh. Eksp. Teor. Fiz.* **83** 1025
[12] Lifshits I M 1960 *Pis. Zh. Eksp. Teor. Fiz.* **38** 1569
[13] Salomaa M M and Volovik G E 1989 *J. Phys.: Condens. Matter* **1** 277